A General Construction of Strict Models in HoTT

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We give a strictification construction that starts from a comprehension category with a universe closed under certain type formers, and constructs a category with families (CwF) which supports the same type formers. This makes it possible to build CwFs from models where the types are not necessarily set-truncated in Homotopy Type Theory (HoTT). Our construction generalises various existing strictification constructions in the literature [17, 9, 7, 2].

1 Motivation

There are various approaches to defining the syntax of type theory within type theory. One successful approach was described by Altenkirch and Kaposi [2], who use quotient inductive-inductive types [11] to construct the initial category with families (CwFs). However, this approach comes with a drawback if ones uses Homotopy Type Theory (HoTT): the notion of CwFs fails to admit interesting models, as it forces the set-truncation of types. For instance, both the set model and presheaf models, which are used to prove normalization by evaluation [3], only give rise to a CwF if one assumes uniqueness of identity proofs, which is incompatible with the univalence axiom.

One possible response is to modify the notion of CwF to allow for more general models. Altenkirch et al. [4] develop the notion of GroupoidCwF, in which types are likewise truncated at the groupoid level. Following this line of work, Damato presents a container model of type theory [7] where types are truncated at the level of groupoids. Another possible approach is to retain the definition of a CwF and instead restrict attention to certain classes of models that give rise to a CwF. For instance, while sets do not give rise to a CwF, one can consider a universe of sets using either iterative sets [9] or induction-recursion [2, 8]. When the model is restricted to such a universe, one indeed obtains a CwF. Note that these constructions only cover specific examples rather than a general instance.

The purpose of our work is to provide a uniform construction that generalises the latter approach. Specifically, we present a strictification construction that allows for building CwF models in which types are not necessarily set-truncated. Our starting point is a full comprehension category with a universe closed under certain type formers. Given such a comprehension category, we construct a corresponding CwF supporting the same type formers. In this way, we obtain CwF models with types that are not necessarily set-truncated.

The choice to use comprehension categories as our starting point is motivated by their generality. As discussed by Ahrens et al. [1], various existing models of type theory embed as sub-2-categories of the 2-category of comprehension categories. Our construction generalises the approach based on universes of iterative sets [9], in that it can be applied to any comprehension category with a universe closed under the required type formers, including presheaf categories, realizability toposes and categories of assemblies. Additionally, our construction is a generalisation of the strictification construction given by Voevodsky [17], which provides a method for strictifying models of type theory in **set-theoretic** foundations. In contrast to this work, our construction is formulated in **type-theoretic** foundations. Our construction also generalises the approach of Altenkirch and Kaposi [2], who use inductive–recursive universes to obtain strict CwF models, where contexts and types are internalised via codes in a universe in order to enforce strict substitution.

It also is important to note the similarity between our construction is and the **externalisation** of internal categories. Specifically, every internal category in a finite complete category $\mathcal{C}$ gives rise to a split (Grothendieck) fibration over $\mathcal{C}$. Since universes give rise to full internal categories, they give rise to a CwF via externalisation. However, note that our construction is formulated in more generality: rather than only considering finitely complete categories, we consider all comprehension categories with a universe.

2 Strictification Construction

In the following, let $(\mathcal{C}, \mathcal{T}, p, \chi)$ be a full comprehension category with a terminal empty context $\diamond : \mathcal{C}$. We use the following type-theoretic notations which are aligned with how the type-theoretical constructs are interpreted in comprehension categories: We use $\mathsf{tm}(A)$ to denote the sections of $\chi(A) : \Gamma.A \to \Gamma$ for $A : \mathcal{T}_\Gamma$. We use $A[s]$ to denote the reindexing of $A : \mathcal{T}_\Delta$ along $s : \Gamma \to \Delta$ in $\mathcal{C}$. Lastly, we denote $\Gamma.a \circ t$ by $t \uparrow_a$ for each $a : A \to A'$ in $\mathcal{T}_\Gamma$ and $t : \mathsf{tm}(A)$, since, as discussed by Najmaei et al. [14], coercing terms t of type A to a term t' of type A' is interpreted in this way.

We begin with the definition of a universe in comprehension categories. We use the definition given by Van der Weide [16, Definition 9.13]. A similar definition can be found in Angiuli and Gratzer's book [5, Structure 6.4.17] for categories with families.

Definition 1 ([16, Definition 9.13])**.** We say $(\mathcal{C}, \mathcal{T}, p, \chi)$ has a universe if we have:

1. An object $U : \mathcal{T}_\diamond$, which gives rise to $U[\Gamma] \coloneqq U[\iota_\Gamma]$ in each $\mathcal{T}_\Gamma$ where $\iota_\Gamma : \Gamma \to \diamond$ is the terminal morphism;

2. For each $\Gamma : \mathcal{C}$, a function $el : \mathsf{tm}(U_\Gamma) \to \mathcal{T}_\Gamma$;

3. For each $s : \Gamma \to \Delta$ and $t : \mathsf{tm}(U_\Delta)$, an isomorphism $i_{s,t} : (\mathsf{el}(t))[s] \cong \mathsf{el}(t[s] \uparrow_e)$ stating that reindexing commutes with el, and making the coherence diagrams with the reindexing isomorphisms commute, where $e : U_{\Delta[s]} \to U_\Gamma$ is given by the universal property of U_Γ.

Note that one can equivalently define el as a type in the context $\diamond.U$. Our function in item 2 can then be obtained by reindexing along the canonical morphism $\Gamma \to \diamond.U$.

In the following construction, we use the usual definitions of unit and sigma types as the fiberwise terminal object and left adjoint to reindexing, respectively.

Construction 2. From a full comprehension category $(\mathcal{C}, \mathcal{T}, p, \chi, \diamond)$ with universe (U, el), where the universe is closed[1] under unit and sigma types, we can construct a category with families as follows:

1. The category of contexts has $\mathsf{tm}(U)$ as objects, and substitutions are given by context morphisms of the form $\diamond.el(\Gamma) \to \diamond.\mathsf{el}(\Delta)$ in the comprehension category. The terminal context of the CwF is then given by the unit type of the comprehension category;

2. Types of the CwF in a context $\Gamma : \mathsf{tm}(U)$ are $\mathsf{tm}(U_{\diamond.\mathsf{el}(\Gamma)})$;

3. Terms of type $A : \mathsf{tm}(U_{\diamond.\mathsf{el}(\Gamma)})$ in a context $\Gamma : \mathsf{tm}(U)$ are $\mathsf{tm}(\mathsf{el}(A))$;

[1] We define the universe being closed under a certain type former similarly to Van der Weide [16].

4. Context extension follows from sigma types in the comprehension category.

The type formers in the strictified model depend on the type formers in the original comprehension category and the closure properties of the universe under these type formers. For example, for the strictified model to support pi-types, the original comprehension category must support pi-types and the universe must be closed under pi-types.

We expect various applications of this constructions. For instance, there is a general construction for universes in presheaves [6, 10, 15], meaning that one can obtain models for presheaves using either induction-recursion or iterative sets. Another application would be in the context of realizability where one can construct universes for assemblies [13] and realizability toposes [12].

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