For Generalised Algebraic Theories, Two Sorts are Enough

Samy Avrillon[1], Ambroise Kaposi[2], Ambrus Lafont[3], Niyousha Najmaei[3], and Johann Rosain[3]

[1] ENS Lyon, Lyon, France
[2] Eötvös Loránd University, Budapest, Hungary
[3] LIX, École polytechnique, Palaiseau, France

We show that any Generalised Algebraic Theory (GAT) can be reduced to a GAT with two sorts, and there is a section-retraction correspondence (a strict coreflection) between models of the original and the reduced GAT. Our approach is semantic; it does not rely on a syntactic description of GATs, but instead, on Uemura's bi-initial characterisation of the category of (finite) GATs in the 2-category of cartesian categories with a chosen exponentiable morphism.

Introduction Generalised algebraic theories (GATs), introduced by Cartmell [1], extend algebraic theories by allowing the specification of multi-sorted algebraic structures in which sorts may be indexed over each other. Typical examples include the GAT of transitive graphs, with sorts of vertices and edges indexed over pairs of vertices, and the GAT of Martin-Löf type theory, whose sorts of contexts, substitutions, types, and terms are mutually indexed. A Russell universe can be specified as a type in any context with a sort equality enforcing that the set of terms of this type is exactly the set of types in Γ. More examples of GATs can be found in [1].

In this work, we show that any GAT can be reduced to a family GAT, that is, a GAT whose only sort declarations are $\mathsf{U} : \mathsf{Set}$ and $\mathsf{El} : \mathsf{U} \to \mathsf{Set}$. We call this reduction *two-sortification*. The translation replaces occurrences of Set with U and inserts El in front of every used sort, and prefixes the resulting theory with the declarations of U and El. We demonstrate two-sortification using the example of the GAT of transitive graphs. This GAT consists of two sort declarations $\mathbf{V} : \mathsf{Set}, \mathbf{E} : \mathbf{V} \to \mathbf{V} \to \mathsf{Set}$, and an operation $\mathbf{T} : \prod v_1, v_2, v_3 : \mathbf{V}.E\, v_1\, v_2 \to E\, v_2\, v_3 \to E\, v_1\, v_3$. Its two-sortified GAT is $\mathbf{U} : \mathsf{Set}, \mathbf{El} : \mathsf{U} \to \mathsf{Set}, \mathbf{V} : \mathsf{U}, \mathbf{E} : \mathsf{El}\,\mathbf{V} \to \mathsf{El}\,\mathbf{V} \to \mathsf{U}, \mathbf{T} : \prod v_1, v_2, v_3 : \mathsf{El}\,\mathbf{V}.\mathsf{El}(E\, v_1\, v_2) \to \mathsf{El}(E\, v_2\, v_3) \to \mathsf{El}(E\, v_1\, v_3)$. Note that no matter how many sorts the original GAT has, the reduced GAT has only two sorts U and El.

A key aspect of this work is that our approach is semantic and does not rely on a specific syntactic presentation of GATs. We use Uemura's [4] bi-initial characterisation of the category of finite GATs in the 2-category of cartesian categories with a chosen exponentiable morphism. We extend this characterisation to include infinite GATs by considering categories with all small limits instead of just finite ones.

Two-sortification yields a GAT that is simpler than the original in the following aspects: it has no sort equalities; it has no interleaving between sort declarations and operations; and if the original GAT does not have interleaved sorts and operations, then the reduced GAT has no operations interleaved between different sorts. In a type-theoretic metatheory, the initial algebra of a GAT is given by a quotient-inductive-inductive type (QIIT). Two-sortification therefore provides a way to implement QIITs with sort equalities or interleaved constructors, which are not supported by Cubical Agda.

Our work also fits in the line of research consisting in constructing initial models by reducing some notion of GATs to a simpler one. Two-sortification can be seen as an extension of the reduction of mutual inductive types to indexed inductive types [2]. Indeed, consider a mutual definition of three inductive types A, B, and C, as the initial model of a generalised algebraic theory $\mathbf{A} : \mathsf{Set}, \mathbf{B} : \mathsf{Set}, \mathbf{C} : \mathsf{Set}, \mathbf{a} : B \to A, \ldots$. Two-sortification yields $\mathbf{U} : \mathsf{Set}, \mathbf{El} : \mathsf{U} \to$

$\mathsf{Set}, \mathbf{A} : \mathsf{U}, \mathbf{B} : \mathsf{U}, \mathbf{C} : \mathsf{U}, \mathbf{a} : \mathsf{El}\, B \to \mathsf{El}\, A, \ldots$. Note that the two sorts U and El are no longer mutual.

The coreflection ensures the existence of the initial model of the original GAT whenever the reduced GAT has one. In fact, Sestini [3] explicitly conjectured the validity of two-sortification for his variant of GATs to justify focusing on the two-sorted ones, and reducing them further to indexed-inductive types, in a type-metatheoretic setting. We confirm this conjecture in a more general setting.

We give a syntactic and a semantic account of two-sortification. Syntactically, we define it as an endofunctor on the category of GATs. Semantically, we show that there is a strict coreflection between the categories of models of a GAT and that of its two-sortification, with the roundtrip starting from the original GAT being the identity. We also characterise the category of models of the reduced GAT in terms of the models of the original GAT equipped with suitable families. These results justify two-sortification as a semantics-preserving translation of GATs.

The Universal Property of GATs We generalise Uemura's [4] bi-initial characterisation of the category of finite GATs to GATs that are not necessarily finite by considering categories with all small limits equipped with an exponentiable morphism, instead of just finite limits. Recall that a morphism $f : Y \to X$ in a cartesian category $\mathbb{C}$ is said to be *exponentiable* if the pullback functor $f^* : \mathbb{C}/Y \to \mathbb{C}/X$ along f has a right adjoint, called the pushforward along f.

Specifically, the category of GATs is bi-initial in the 2-category **ComplExp** of complete categories equipped with an exponentiable morphism, and continuous functors between them preserving the exponentiable morphisms and pushforwards along it. Intuitively, the category of GATs is the category of (possibly infinite) contexts and substitutions of the type theory generated by a universe Set of types, extensional equality types, and dependent products over types in Set. The exponentiable morphism is then the projection $(\mathbf{A} : \mathsf{Set}, \mathbf{a} : A) \to (\mathbf{A} : \mathsf{Set})$. The associated polynomial functor P associated maps a GAT to the same GAT but parameterised by a fresh sort X, inserted at the beginning. For example, $P(\mathbf{A} : \mathsf{Set}, \mathbf{a} : A)$ is $(\mathbf{X} : \mathsf{Set}, \mathbf{A} : X \to \mathsf{Set}, \mathbf{a} : \prod(x : X), A\, x)$.

Let us relate this abstract characterisation with context extension of GATs. Say we want to extend a GAT with a sort with two dependencies, i.e., a sort declaration of the shape $\prod(a : A), B\, a \to \mathsf{Set}$, in the most general case. The required data to specify such a sort in a theory Γ consists of a sort A and a sort B indexed over A in Γ. This precisely corresponds to the data of a morphism from Γ to $(\mathbf{A} : \mathsf{Set}, \mathbf{B} : A \to \mathsf{Set})$. The extended context is then the pullback of this morphism along the projection from $(\mathbf{A} : \mathsf{Set}, \mathbf{B} : A \to \mathsf{Set}, \mathbf{C} : \prod(a : A), B\, a \to \mathsf{Set})$ to $(\mathbf{A} : \mathsf{Set}, \mathbf{B} : A \to \mathsf{Set})$. In a sense, this projection is thus the universal context extension by a sort with two dependencies. In fact, this projection is nothing but the morphism $PP(\mathbf{A} : \mathsf{Set}) \to P(\mathbf{A} : \mathsf{Set})$ obtained by applying P to the canonical projection $P(\mathbf{A} : \mathsf{Set}) = (\mathbf{X} : \mathsf{Set}, \mathbf{A} : X \to \mathsf{Set}) \to (\mathbf{X} : \mathsf{Set}) \cong (\mathbf{A} : \mathsf{Set})$. More generally, the canonical morphism $P^{n+1}(\mathbf{A} : \mathsf{Set}) \to P^n(\mathbf{A} : \mathsf{Set})$ is the universal context extension by a sort with $n+1$ dependencies. A similar reasoning applies to context extension by an operation: the universal constructor with n arguments is $P^n(\mathbf{A} : \mathsf{Set}, \mathbf{a} : A) \to P^n(\mathbf{A} : \mathsf{Set})$.

The Syntactic Translation We define the *two-sortification functor* $T : \mathbf{Gat} \to \mathbf{Gat}/\mathbf{Fam}$ as an *initial functor* mapping a GAT to its translation equipped with its projection to the GAT of families. The key point in defining the two-sortification functor is to equip the slice category $\mathbf{Gat}/\mathbf{Fam}$ with a suitable exponentiable morphism. Additionally, we show that the two-sortification functor is fully faithful.

Theorem 1 (Soundness)**.** *The image of any GAT by the two-sortification functor is isomorphic to a GAT whose only two sorts are* U *and* El*.*

The Semantic Translation and Models of the Reduced GAT We give a description of the category of models of the two-sortification of a GAT using an initial **ComplExp**-functor $[\![-]\!] : \mathbf{Gat} \to \mathbf{Cat}$ computing the category of models of a given GAT. For this, we first define, for every theory Γ, a *family functor* $[\![\Gamma]\!] \to \mathbf{Fam}$, using the bi-initiality of **Gat** in **ComplExp**. More specifically, we equip the colax slice category $\mathbf{Cat} /\!\!/ \mathbf{Fam}$ with a suitable exponentiable morphism, and use the universal property to obtain for each theory Γ, a functor from its category of models to **Fam**. It is worth noting that intuitively, this family functor is the composition of the coreflective left adjoint $[\![\Gamma]\!] \to [\![T\Gamma]\!]$ and a projection from $[\![T\Gamma]\!]$ to **Fam**. Such a projection exists since $[\![T\Gamma]\!]$ is always a GAT over **Fam**. We denote the image of a model M under its family functor by (U_M, El_M). The following theorem characterises the category $[\![T\Gamma]\!]$.

Theorem 2. *The category* $[\![T\Gamma]\!]$ *of models of the two-sortification of* Γ *is isomorphic to the category where an object is a model* M *of the original GAT, a family* (U', El')*, and a function* $f : U_M \to U'$ *such that* $El'(f(u)) = El_M(u)$ *for each* $u \in U_M$*, and a morphism between two objects consists of a morphism between the underlying models and a morphism between the underlying families compatible with the underlying functions.*

As a corollary, for any GAT Γ, there is a strict coreflection between $[\![\Gamma]\!]$ and $[\![T\Gamma]\!]$. Under the above isomorphism, the right adjoint forgets the family structure, while the left adjoint embeds a model via the identity family; hence, the roundtrip is the identity.

Additionally, we show that each theory Γ has an initial model, using the bi-initial characterisation above mentioned.

An anonymous referee suggested that our results can be recovered from a combination of more general conservativity statements about SOGATs and representable map categories, including conservativity of two-level type theory. We hope to investigate this connection as well.

References

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